Introduction to (big) Cohen-Macaulay modules

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Big CMinar, January 31st, 2025

These notes mostly follow Bruns and Herzog [2] and occasionally Mel's notes `https://dept.math.lsa.umich.edu/~hochster/615W20/615.html` [4]. There are also some exercises, of examples/proofs/explanations I had to cut!

* exercise = I thought it is more useful/informative than the others.
† exercise = there are some hints in Section 6.

1 Regular sequences & depth

Definition 1.1. Let M be an R-module A *regular sequence* on M is a sequence of ring elements $x_1, \ldots, x_n =: \underline{x}$ such that both:

1. For all $1 \leq i \leq n$, x_i is a non-zerodivisor on $M/\langle x_1, \ldots, x_{i-1}\rangle M$; and
2. $\langle x_1, \ldots, x_n\rangle M \neq M$

It is a *possibly improper regular sequence* if we remove the second condition.

[[Taking $i = 1$, this means we need x_1 to be a non-zero divisor on M.]]

Other terminology you might see in the wild:

- Could call $x_1, \ldots, x_n$ an M-regular sequence, or just an M-sequence
- Could say M is $\underline{x}$-regular

Example 1.2. Some examples:

- $R = M = k[x_1, \ldots, x_n]$ then the x_i are a regular sequence
- $R = k[x, y]/\langle x^2, xy\rangle$ and $M = \langle y\rangle$. Then y is a regular sequence.
- $R = M = k[x, y]$ then $x,\ y - xy,\ z - xz$ is a regular sequence

Example 1.3. Some NON-examples:

- $R = M = k[s^4, s^3t, st^3, t^4] \subset k[s, t]$, then s^4, t^4 is NOT a regular sequence.
- $R = M = k[x, y]$, then $y - xy,\ z - xz,\ x$ is NOT a regular sequence.

Fact 1.4. *If $(R, \mathfrak{m})$ is a local ring and M is finitely generated, then any regular sequence is permutable. (Also holds if R is standard graded and the elements in the sequence are all homogeneous).*

See Section 5.1 for an example of a non-f.g. module where permutability fails.

Fact 1.5. *Let $(R, \mathfrak{m})$ local and M finitely generated. Then all maximal M-regular sequences have the same length.*

Example 1.6 (Courtesy of Olivia Strahan). Even for a noetherian local ring, if M is not f.g., it is possible to have two maximal M-sequences of different length. Let $R = k[[x, y_1, y_2]]$ and let

$$N = \langle y_1 u_i - x u_{2i-1},\ y_2 u_i - x u_{2i} \mid i \in \mathbb{N} \rangle \subset \bigoplus_{i \in \mathbb{N}} R u_i, \qquad M = \frac{\bigoplus_{i \in \mathbb{N}} R u_i}{N}$$

so that $xu_1 = y_1 u_1$, $xu_3 = y_1 u_2$, $xu_5 = y_1 u_3$ etc and similarly for $xu_2 = y_2 u_1$, $xu_4 = y_2 u_2$, etc. Now x is a maximal regular sequence since it is a nzd and

$$M/xM \cong \frac{\bigoplus_i R u_i}{\langle x u_i, y_1 u_i, y_2 u_i \rangle_{i \in \mathbb{N}}} \cong \bigoplus_i (R/\langle x, y_1, y_2 \rangle) u_i.$$

But y_1, y_2 is a longer regular sequence, since

$$M/y_1 M \cong \frac{\bigoplus_i R u_i}{\langle y_1 u_i,\ x u_{2i-1},\ y_2 u_i - x u_{2i} \rangle_{i \in \mathbb{N}}}$$

and this means every element of the quotient has a representative of the form $\sum_i f_i u_i$ where all of the $f_i \in k[[y_2]]$. [[Since ordinarily would know $f_i \in R$, but also $y_1 u_i = 0$, $x u_{2i-1} = 0$, and $x u_{2i} = y_2 u_i$.]] So y_2 nzd on this quotient.

Definition 1.7. Let $(R, \mathfrak{m})$ noetherian local and M a finitely generated R-module. The *depth* of M is the length of a maximal regular sequence on M contained in $\mathfrak{m}$. [[Can also consider the I-depth, written $\operatorname{depth}_I M$, where now we only consider M-sequences contained in I.]]

Theorem 1.8. *If $(R, \mathfrak{m}, k)$ noetherian local and M finitely generated, then*

$$\operatorname{depth}_R M = \inf\{ i \in \mathbb{N} \mid \operatorname{Ext}^i_R(k, M) \neq 0 \}.$$

[[We're taking convention that $\inf \emptyset = +\infty$, so in particular $\operatorname{depth} 0 = \infty$. But I'm generally going to be a bit sloppy about the zero module.]]

Example 1.9 (From Strooker). Even for a noetherian local ring, if M is not finitely generated, it's possible to have depth (in the regular sequence sense) less than this "Ext-depth". Let $R = k[[x, y]]$ and

$$M = \bigoplus_{\substack{0 \neq r \in R \\ r \notin R^\times}} R/\langle r \rangle.$$

Then every element of R is a zero-divisor, so depth $= 0$. However, one can check that "Ext-depth"=1. See [7, Sec. 5.3] for more info (and p.92 for this example).

Definition 1.10. $\dim M = \dim \operatorname{Supp} M$, taking the dimension as a topological space inside of $\operatorname{Spec} R$ (so, the maximum length of a chain of irreducible subvarieties). Recall that $\operatorname{Supp} M = \{ \mathfrak{p} \in \operatorname{Spec} R \mid M_\mathfrak{p} \neq 0 \}$

Remark 1.11. A correction from the talk: If M is finitely generated, then it is also true $\dim M = \dim(R/\operatorname{ann} M)$, since $\mathbb{V}(\operatorname{ann} M) = \operatorname{Supp}(M)$ in this case. However, if M is not f.g., these are NOT the same and we only have that $\operatorname{Supp}(M) \subset \mathbb{V}(\operatorname{ann} M)$ and that $\dim(R/\operatorname{ann} M) \leq \dim M$. Example: If $R = k[[x]]$ and $M = H^1_{\langle x \rangle}(R) = k[[x]][x^{-1}]/k[[x]] \cong \bigoplus_{t=1}^\infty k \cdot x^{-t}$, then $\operatorname{ann}_R M = 0$ and so $\dim(R/\operatorname{ann} M) = \dim R = 1$. But every element of M is torsion, so any further localization will kill M and thus $\operatorname{Supp} M = \{\langle x \rangle\}$, which means $\dim \operatorname{Supp}(M) = 0$.

Theorem 1.12. $\operatorname{depth}_R M \leq \dim M$. *[[Even if M not f.g., this is still true when phrased as "any M-regular sequence has length at most $\dim(R/\operatorname{ann} M)$"]]*

Proof sketch. x_1 non-zero divisor on $M \implies x_1$ not in any minimal prime of $\operatorname{ann} M$. Therefore $\dim_R M - 1 = \dim_{R/r_1} M$. Now induct. □

1.1 Exercises

1. * Prove that everything in Example 1.2 is a regular sequence; and prove that everything in Example 1.3 is NOT a regular sequence.
2. * † Prove that if $x_1, \ldots, x_n$ is a regular sequence, then so is $y_1, \ldots, y_n$ where $y_i = x_i + \sum_{j=1}^{i-1} c_{i,j} x_j$ for any collection of ring elements $c_{i,j}$.
3. Prove that if $x_1, \ldots, x_n$ is a regular sequence, then so is $x_1^{a_1}, \ldots, x_n^{a_n}$ for any set of positive integers a_i.
4. * † Prove that regular sequences are permutable in local rings.
5. † (Ex 1.2.20 in [2]) Let $R = k[[x]][y]$. Show that x, y is a maximal regular sequence. Show that $1 - xy$ is a maximal regular sequence. Why doesn't this contradict Fact 1.5?
6. Using the same idea as in Remark 1.11, for every n find an example of a module M such that $\dim \operatorname{Supp} M = 0$ but $\dim(R/\operatorname{ann} M) = n$.

2 Cohen-Macaulay rings & modules

Definition 2.1. Let $(R, \mathfrak{m})$ be noetherian local. A finitely generated R-module M is *Cohen-Macaulay* if $\operatorname{depth}_R M = \dim M$. A ring R is Cohen-Macaulay if it is CM as an R-module, i.e., if $\operatorname{depth}_R R = \dim R$.

[[By the fact in section above, we only need to check that $\dim M \leq \operatorname{depth}_R M$! Or in other words, we only need to find some regular sequence whose length is $\dim M$.]]

Example 2.2. Some examples of CM rings:

- A polynomial ring over a field: $k[x_1, \ldots, x_d]$ has $\dim = d$, and has regular sequence $x_1, \ldots, x_d$
- Any regular ring
- Any zero-dimensional ring.
- Any one-dimensional reduced ring. [[only need a single nzd. Reduced $\implies$ all zero-divisors are contained in minimal primes. Since 1-dim, can find some element NOT in a minimal prime]]
- Any two-dimensional normal ring.

Definition 2.3. A *system of parameters* (or *s.o.p.*) of local ring $(R, \mathfrak{m})$ is a set of $d = \dim R$ elements $x_1, \ldots, x_d$ such that any of TFAE hold:

- $\sqrt{\langle x_1, \ldots, x_d \rangle} = \mathfrak{m}$.
- The ideal $\langle x_1, \ldots, x_d \rangle$ is $\mathfrak{m}$-primary.
- The ring $R/\langle x_1, \ldots, x_d \rangle$ has Krull dimension $= 0$.

These always exist! Choose x_1 not in any minimal prime, then x_2 a lift of something not in any minimal prime of R/x_1, etc. Dimension drops by 1 each time, so result has d elements whose quotient is Krull dimension zero.

[[We see an intuitive connection here for f.g. M: To build a regular sequence, we're choosing something not in any *associated* prime of the quotient. So if a reg seq has length d, it automatically is an s.o.p.]]

Theorem 2.4. *Let $(R, \mathfrak{m})$ be noetherian local. Let M be a finitely generated R-module. Then TFAE:*

1. depth $M = \dim R$*, i.e., M is CM and has* $\dim M = \dim R$*.*
2. *Some s.o.p. of R is a regular sequence on M.*
3. *Every s.o.p. of R is a regular sequence on M.*

Call such an M maximal CM *or* small CM*.*

Notice: any ring is in fact maximal CM! So for rings, always equivalent to think about some/every s.o.p.

[[Contradictory sounding names? The maximal is referring to the fact that the Krull dimension is biggest possible for this ring. The small is opposed to...]]

Definition 2.5. Let $(R, \mathfrak{m})$ be noetherian local, and let M be any R-module. M is *big Cohen-Macaulay* if some s.o.p. of R is a regular sequence on M. Say M is *balanced* big CM if EVERY s.o.p. is M-regular.

Example 2.6. Some examples of maximal CM modules:

- Any ring that is CM is automatically maximal.
- $R = k[x, y]/\langle x^2, xy\rangle$ and $M = \langle y\rangle$

Example 2.7. Some NON-examples:

- $R = M = k[x, y]/\langle x^2, xy\rangle$. [[in particular, this is a ring which is not CM over itself, but we see from the above that there nonetheless exists a maximal CM module for it]]
- $R = M = k[s^4, s^3t, st^3, t^4]$.

What about non-local rings?

Definition 2.8. If R is noetherian and M is finitely generated, then say M is CM if $M_\mathfrak{p}$ is CM as an $R_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R (or equivalently, for every maximal ideal).

[[There is something to check here! It's not obvious that its equivalent to only check maximal ideals. But this equivalence is what makes this a "good" definition, because it ensures that if R is local, Definition 2.1 and Definition 2.8 agree.]]

Theorem 2.9. *Let R be a finitely generated $\mathbb{N}$-graded K-algebra with $R_0 = K$. Then R is CM if and only if some homogeneous s.o.p. is a regular sequence.*

Proof. In [4], use the proposition on page 113, and combine with the theorem on page 9 in order to see that the F_i's that appeared actually come from a homogeneous s.o.p. in particular. □

2.1 Exercises

1. † Prove that two dimensional local normal rings are CM. (Recall that a local ring is *normal* if it is a domain which is integrally closed in its field of fractions.)

2. * † Prove that all the definitions of an s.o.p. in Definition 2.3 are equivalent.

3. * † Confirm the examples/nonexamples from Example 2.6 and Example 2.7.

3 Čech complex

Consider a sequence $x_1, \ldots, x_n$ of ring elements we can build the *Čech complex* via

$$\check{C}^{\bullet}(\underline{x}; R) = 0 \longrightarrow R \xrightarrow{d_0} \bigoplus_{i=1}^{n} R_{x_i} \xrightarrow{d_1} \bigoplus_{i \neq j} R_{x_i x_j} \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} R_{x_1 \cdots x_n} \longrightarrow 0$$

where the restriction of d_i to $R_{x_{j_1} \cdots x_{j_i}}$ is the localization (with some alternating negative signs).

Example 3.1. The Čech complex on two elements x, y is

$$0 \longrightarrow R \longrightarrow R_x \oplus R_y \longrightarrow R_{xy} \longrightarrow 0$$

$$1 \longrightarrow (1,1)$$

$$(1,0) \longrightarrow -1$$

$$(0,1) \longrightarrow 1$$

For a module M,let $\check{C}^{\bullet}(\underline{x}; M) := \check{C}^{\bullet}(\underline{x}; R) \otimes_R M$.

Definition 3.2. Let $I = \langle x_1, \ldots, x_n \rangle$ and M an R-module. Then the i-th local cohomology is

$$H_I^i(M) = H^i(\check{C}(\underline{x}; M)).$$

Fact 3.3. *This is independent of the choice of generators for I. In fact, if $\sqrt{I} = \sqrt{J}$, then $H_I^i(M) = H_J^i(M)$ for all i and M.*

Theorem 3.4 ([6, Thm. 10.36])**.** *Let $(R, \mathfrak{m})$ be a local ring, and let M be an f.g. R-module. Then M is Cohen-Macaulay if and only if $H_{\mathfrak{m}}^i(M) = 0$ for all $i \neq \dim M$.*

In fact, as it turns out we have the following:

Theorem 3.5 ([6, Thm. 9.1])**.** *If R noetherian, $\mathfrak{a}$ an ideal, $K^{\bullet}$ is the Koszul complex on a finite generating set for $\mathfrak{a}$, and M an R-module, then*

$$\inf\{n \mid \operatorname{Ext}_R^n(R/\mathfrak{a}, M) \neq 0\} = \inf\{n \mid H_{\mathfrak{a}}^n(M) \neq 0\} = \inf\{n \mid H^n(\hom_R(K^{\bullet}, M) \neq 0\}.$$

So when M is finitely generated, all of these equal the $\mathfrak{a}$-depth of M (where the regular sequence is forced to be contained in $\mathfrak{a}$).

4 Miracle Flatness

Theorem 4.1 ([6, Thm. 10.13])**.** *Let K be a field, R be an $\mathbb{N}$-graded ring f.g. over $R_0 = K$. Let $x_1, \dots, x_d$ be a homogeneous s.o.p. Then R is CM if and only if R is free over $K[x_1, \dots, x_d]$*

Or, a local version:

Theorem 4.2. *Let $(R, \mathfrak{m})$ be noetherian local, and let $(S, \mathfrak{n})$ be a regular local ring inside R such that $S \to R$ is module finite. Then R is CM if and only if it is free over S.*

The connection: Notice that in the first statement, by design the ring $K[x_1, \dots, x_d]$ is regular, and R is a module finite extension of it.

[[Why the name? For "flatness": Remember for f.g. modules over noetherian local rings, flat and free are the same! For "miracle": its straightforward to show that if R is free over such an S, then R is CM. The miracle is that the other direction also works.]]

5 Big CM modules

Not all is bad!

Theorem 5.1 ([5, Thm. A])**.** *Let R be a local CM ring with a dualizing module (such as a complete local CM ring). Every balanced big CM module M "comes from" small CM modules. More specifically, M is a direct limit of small CM modules.*

We can also use the local cohomology idea from above.

Definition 5.2. Let S be a noetherian ring (of finite Krull dimension) and M be an S-module. Then M is *cohomologically CM* if $H^i_{\mathfrak{p}}(M_{\mathfrak{p}}) = 0$ for all primes $\mathfrak{p}$ and for all $i < \operatorname{ht} \mathfrak{p}$.

Theorem 5.3 ([1, Cor. 2.8], From cohomological CMness to regular sequences)**.** *If S is a catenary and equidimensional noetherian local ring and M is an S-module which is cohomologically CM, then every system of parameters on S is a regular sequence on M, i.e., M is a balanced big CM module.*

5.1 An extended example

Example is from Griffith, see [3, Rmk. 3.3] and [5]. Let $R = k[[x, y]]$, so $\dim R = 2$. Let $N = R_{\langle y \rangle}$ and $M = R \oplus (N_y/N)$. [[So $N_y/N = E(R/y)$ is the injective hull.]] We'll show M is big CM but not balanced. [[All of the interesting stuff happens in the second component; the 1st piece is just to make sure nothing is improper.]]

Take s.o.p. of x, y. We see multiplication $\cdot x : N_y/N \to N_y/N$ is an isomorphism, because $x \notin \langle y \rangle$ and so acts as a unit on N. Clearly nzd on 1st component. Now $M/xM \cong R/xR$, and again y nzd on 1st (only) component. Therefore we're big CM.

Now take s.o.p. of y, x. [[remember permutability was only for f.g. modules!]] We claim this is not regular, because y is a zero-divisor. Elements of N_y are of the form $\frac{n}{y^t}$ for $t \in \mathbb{N}$, $n \in N$. But $y \cdot \frac{n}{y} \in N$ and so y annihilates a lot of things. Therefore we're not balanced.

(Not) Cohomological CM: One can check that just taking local cohomology at $\mathfrak{m}$ "works" (i.e., appears to get zero in all the right places) but the problem is that we need to check for all primes. This doesn't localize! So this example explains the need for the "$\forall \mathfrak{p}$" in the definition of cohomologically CM.

6 Hints to selected exercises

Hints to 1.1

For #2: Think about the quotient rings.

For #4: Hint 1: show we can reduce to the case of checking that we can swap the elements of a length two regular sequence. (Hint 1.5: transpositions?)

Hint 2: Apply NAK. (This also tells you more precisely the right hypotheses for the graded version of this statement— R is $\mathbb{N}$-graded with R_0 a field, M is $\mathbb{Z}$-graded with $M_i = 0$ for all $i \ll 0$, the elements in R are homogeneous of positive degree).

For #5: For showing $1 - xy$ is maximal, show the quotient ring $R/\langle 1 - xy \rangle \cong k[[x]][x^{-1}]$ and think about prime ideals in a localization (or: think about the units).

Hints to 2.1

For #1: It's a domain so the first non-zerodivisor (call it x) is easy. Go by contradiction for the 2nd—if you can't get a 2nd, show that this means $\exists r \neq 0 \in R$ s.t. $\mathfrak{m} \cdot r = 0 \in R/\langle x \rangle$. Then show that r/x is integral over R and use this to get contradiction.

For #2: For 1st to 2nd, think about the minimal primary decomposition.

For #3: Use the examples for the 1st section. And for the affine semigroup ring, focus on showing the s.o.p. s^4, t^4 is not regular. Use the fact that s^2t^2 is NOT in the ring.

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