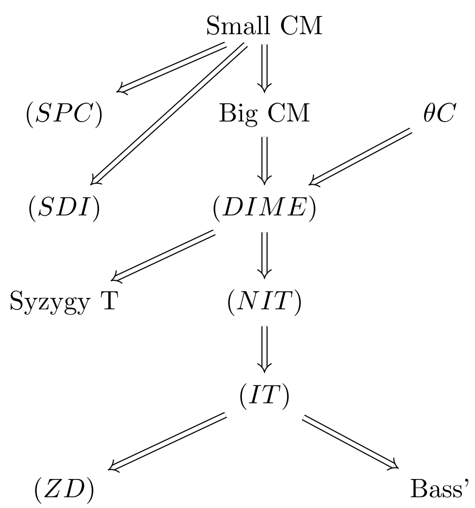
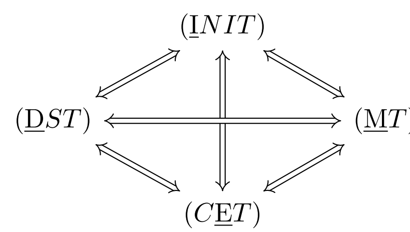
Survey of the homological conjectures, by Mark E. Walker

1. A diagram of homological conjectures

Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring of Krull dimension d. The goal of this 50-minute talk is to prove all of the implications in the following diagrams; there will be, needless to say, little time for questions.



where $(DIME) =$



Key:

- **Small CM**: The conjecture that R has a (finitely generated) MCM module
- **Big CM**: The theorem that R has a big MCM module
- **(DIME)**: A family of four equivalent (now) theorems:
  - **DST**: Direct Summand Theorem
  - **INIT**: Improved New Intersection Theorem
  - **MT**: Monomial Theorem
  - **CET**: Canonical Element Theorem
- **NIT**: New Intersection Theorem
- **IT**: Intersection Theorem of Peskine and Szpiro
- **ZD**: Zero Divisor Theorem
- **SDI**: Serre's Dimension Inequality
- **Bass**: Bass' Conjecture/Theorem
- **SPC**: Serre's Positivity Conjecture for intersection multiplicity
- θ **C**: Hochster's theta conjecture (obscure)
- **Syzygy T**: Syzygy Theorem of Evans' and Griffith's

Precise statements: Recall $(R, \mathfrak{m}, k)$ is a local ring of dimension d.

Conjecture 1.1 (Small CM). *R admits a (non-zero, finitely generated) maximal Cohen-Macaulay module.*

Theorem 1.2 (Big CM). *R admits a big maximal Cohen-Macaulay module.*

Theorem 1.3 (DST: Direct Summand Theorem). *If $\iota : R \subseteq A$ is a module-finite ring extension with R regular, there is an R-module homomorphism $p : A \to R$ such that $p \circ \iota = \mathrm{id}_R$ (and hence $A = R \oplus \ker(p)$ as an R-module).*

Theorem 1.4 (MT: Monomial Theorem). *For any system of parameters $x_1, \dots, x_d$ we have*

$$x_1^t \cdots x_d^t \notin (x_1^{t+1}, \dots, x_d^{t+1})R, \text{ for all } t > 0.$$

Theorem 1.5 (INIT: Improved New Intersection Theorem). *If*

$$F_\bullet = (0 \to F_s \to \cdots \to F_0 \to 0)$$

is a finite free complex of R-modules such that $\mathrm{length}_R\, H_i(F_\bullet) < \infty$ for $i \geq 1$ and there is $\alpha \in H_0(F_\bullet) \setminus \mathfrak{m}H_0(F_\bullet)$ such that $\mathfrak{m}^N \alpha = 0$ for $N \gg 0$, then $s \geq \dim(R)$.

Theorem 1.6 (CET: Canonical Element Theorem). *The canonical map $\mathrm{Ext}_R^d(k, \Omega^d k) \to H_{\mathfrak{m}}^d(\Omega^d k)$ is non-zero, where $\Omega^d k$ denotes the d-th syzygy of k.*

Theorem 1.7 (SDI: Serre Dimension Inequality). *Let R be a regular ring, $\mathfrak{p}$ and $\mathfrak{q}$ prime ideals such that $\mathrm{supp}(R/\mathfrak{p}) \cap \mathrm{supp}(R/\mathfrak{q}) = \{\mathfrak{m}\}$ (equivalently, $R/\mathfrak{p} \otimes_R R/\mathfrak{q}$ has finite length). Then $\dim \mathrm{supp}(R/\mathfrak{p}) + \dim \mathrm{supp}(R/\mathfrak{q}) \leq \dim(R)$.*

Conjecture 1.8 (SPC: Serre Positivity Conjecture). *Let R, $\mathfrak{p}$ and $\mathfrak{q}$ be as in SDIC and assume also that $\dim \mathrm{supp}(R/\mathfrak{p}) + \dim \mathrm{supp}(R/\mathfrak{q}) = \dim(R)$. Then the intersection multiplicity $\chi(R/\mathfrak{p}, R/\mathfrak{q}) := \sum_i (-1)^i \mathrm{length}_R \mathrm{Tor}_R^i(R/\mathfrak{p}, R/\mathfrak{q})$ is strictly positive.*

Theorem 1.9 (NI: New Intersection Theorem). *If $F_\bullet = (0 \to F_s \to \cdots \to F_0 \to 0)$ is a finite free complex of R-modules with finite length homology and $H_0(F_\bullet) \neq 0$ then $s \geq \dim(R)$.*

Conjecture 1.10 (θ C: Hochster's θ Conjecture). *Let V be an unramified mixed characteristic dvr of residue characteristic $p > 0$. For the ring*

$$R = V[x_1, \dots, x_n, y_1, \dots, y_n]/(x_1^t \cdots x_n^t - \sum_i y_i x_i^{t+1}), \quad \text{with } t \geq 2,$$

we have $\theta_R(N, R/(x_1, \dots, x_n)) = 0$ for all N of finite projective dimension on $\mathrm{Spec}(R) \setminus V(p, x_1, \dots, x_n, y_1, \dots, y_n)$. (Here, $\theta_R(N, M) = \mathrm{length}\, \mathrm{Tor}_{2i}^R(N, M) - \mathrm{length}\, \mathrm{Tor}_{2i+1}^R(N, M)$ for $i \gg 0$, when defined.)

Theorem 1.11 (Syzygy T: Evans' and Griffith's Syzygy Theorem). *If L is a k-th syzygy, has finite projective dimension, and is not free, then $\mathrm{rank}_R(L) \geq k$.*

Theorem 1.12 (IT: Intersection Theorem of Peskine and Szpiro). *If $M \otimes_R N$ has finite length for non-zero finitely generated modules M and N, then $\dim(N) \leq \mathrm{pd}_R(M)$.*

Theorem 1.13 (ZD: Zero Divisor Theorem). *If $M \neq 0$ is finitely generated and has finite projective dimension and $r \in R$ is a non-zero-divisor on M, then r is a non-zero-divisor on R.*

Theorem 1.14 (Bass'). *If R admits a non-zero, finitely generated module of finite injective dimension, then R must be Cohen-Macaulay.*

2. Introduction

The goal of this talk is to explain the significance of the existence of small and big MCM modules, with a particular focus on "intersection theorems". In particular, I will show (with varying degree of rigor) every implication in the chain

$$\text{Small CM} \Longrightarrow \text{Big CM} \Longrightarrow (DIME) \Longrightarrow NIT \Longrightarrow IT$$

as well as

$$\text{Small CM} \Longrightarrow (SDI) \text{ and } \text{Small CM} \Longrightarrow (SPC)$$

For the purposes of this talk we will take the four equivalent statements comprising $(DIME)$ to refer to $(INIT)$. Tom will talk about $(DIME)$ next week.

Throughout, $(R, \mathfrak{m}, k)$ is a local commutative ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. The letters M, N, etc. will denote finitely generated R-modules. We'll write $\mathcal{B}$ for a Big MCM R-module.

3. Intersection Theorem

Theorem 3.1 (Peskine-Szpiro Intersection Theorem). *If* $\operatorname{length}_R(M \otimes_R N) < \infty$ *then* $\dim(M) \leq \operatorname{pd}_R(N)$.

Remark 3.2. This was proven by Peskine-Szpiro in 1969. We will show how it follows from other, harder conjectures.

Why "intersection"? Say R is regular, take $M = R/\mathfrak{p}$ and $N = R/\mathfrak{q}$ such that $\operatorname{supp}(R/\mathfrak{p}) \cap \operatorname{supp}(R/\mathfrak{q}) = \{\mathfrak{m}\}$ or, equivalently, $\operatorname{length}_R(R/\mathfrak{p} \otimes_R R/\mathfrak{p})$ is finite. Assume in addition that $R/\mathfrak{q}$ happens to be Cohen-Macaulay.

Since R is regular, $\operatorname{pd}_R(R/\mathfrak{q}) < \infty$ and so by Auslander-Buchsaum $\operatorname{pd}_R(N) = \operatorname{depth}(R) - \operatorname{depth}(R/\mathfrak{q})$ and since R is CM and we assume $R/\mathfrak{q}$ is CM, this gives $\operatorname{pd}_R(N) = \dim(R) - \dim(R/q)$. So, the Intersection Theorem implies in this case that

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R). \tag{3.3}$$

Geometrically, if two affine varieties meet only at a point, the sum of their dimensions cannot exceed the dimension of the ambient affine space. For instance, while a curve and a surface can meet at a single point in $\mathbb{A}^3$, two surfaces in $\mathbb{A}^3$ cannot meet at just one point. Equation (3.3) is a special case of Serre's dimension inequality — special because we assumed $R/\mathfrak{q}$ was CM. Soon we'll prove this in general using the conjectural existence of Small CM modules.

But first let's sketch the proof that $(IT) \Rightarrow (ZD)$.

Theorem 3.4 ((IT) implies (ZD)). *Assume (IT) holds (which it does). If M is an R-module of finite projective dimension, then for every $\mathfrak{p} \in \operatorname{Ass}(R)$, there is a $\mathfrak{q} \in \operatorname{Ass}(M)$ such that $\mathfrak{q} \supseteq \mathfrak{p}$. In particular, (ZD) holds.*

Sketch of Proof. The second assertion follows from the first since non-zero-divisors are those elements not in any associated prime.

For the first assertion, by using induction on $\dim(R)$, it is not hard to reduce to the case where $\operatorname{supp}(R/\mathfrak{p}) \cap \operatorname{supp}(M) = \{\mathfrak{m}\}$ or, equivalently, $\operatorname{length}_R(R/\mathfrak{p} \otimes_R M) < \infty$. In this case, (IT) implies $\dim(R/\mathfrak{p}) \leq \operatorname{pd}_R(M)$. But since $\mathfrak{p} \in \operatorname{Ass}(R)$, we have $\operatorname{depth}(R) \leq \dim(R/\mathfrak{p})$ and thus $\operatorname{depth}(R) \leq \operatorname{pd}_R(M)$. The Auslander-Buchsbaum formula then gives that $\operatorname{depth}_R(M)$ must be zero, and hence $\mathfrak{m} \in \operatorname{Ass}(M)$. $\square$

4. The power of Small CM modules

4.1. **Serre dimension inequality.** Let's illustrate the power of small MCM modules by showing how to deduce (3.3) in general:

Proposition 4.1 (Small CM $\Rightarrow$ SDI). *If R is regular,* $\mathrm{supp}(R/\mathfrak{p}) \cap \mathrm{supp}(R/\mathfrak{q}) = \{\mathfrak{m}\}$ *and $R/\mathfrak{q}$ has a (small) MCM module N, then* (3.3) *holds.*

Proof. Since $\mathrm{supp}(N) = \mathrm{supp}(R/\mathfrak{q})$, we have $\mathrm{length}(R/\mathfrak{p} \otimes_R N) < \infty$ and so by the IT

$$\dim(R/\mathfrak{p}) \leq \mathrm{pd}_R(N) = \mathrm{depth}(R) - \mathrm{depth}(N) = \dim(R) - \dim(N) = \dim(R) - \dim(R/\mathfrak{p})$$

whence (3.3). □

Remark 4.2. Serre's Dimension Inequality was in fact proven by Serre himself, long before these homological conjectures were created. The point of the previous Proposition is just to indicate the power of Small CM modules.

4.2. **Serre positivity.** Let's also show how the existence of Small CM's settles a still open conjecture.

For a regular ring R, if $\mathrm{length}(M \otimes_R N) < \infty$ (equivalently, $\mathrm{supp}(M) \cap \mathrm{supp}(N) = \{\mathfrak{m}\}$) we define $\chi(M, N) = \sum_i (-1)^i \mathrm{length}_R \mathrm{Tor}_i^R(M, N)$. This should be seen as a intersection multiplicity — e.g., for two curves in the plane meeting at the origin, it counts the multiplicity of their intersection at that point. For instance, if $R = k[[x, y]]$, $M = R/f(x, y)$ and $N = R/g(x, y)$ with f and g sharing no common factors, we have $\chi(M, N) = \mathrm{length}\, R/(f, g)$ (since the higher Tor's vanish). For instance, $\chi(k[[x, y]]/(y - x^2), k[[x, y]]/(y + x^2)) = 2$ (if $\mathrm{char}(k) \neq 2$). But in general you need the higher Tor's for χ to be a reasonable invariant.

Conjecture 4.3 (SPC: Serre Positivity Conjecture). *For a regular ring R, if* $\mathrm{supp}(R/\mathfrak{p}) \cap \mathrm{supp}(R/\mathfrak{q}) = \{\mathfrak{m}\}$ *and* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$ *then* $\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0$.

This remains an open conjecture in mixed characteristic.

Proposition 4.4 (Small CM $\Rightarrow$ SPC). *SPC holds provided both $R/\mathfrak{p}$ and $R/\mathfrak{q}$ admit small MCM modules M and N.*

Sketch of Proof. The result follows immediately from the following two facts:

1. $\chi(M, N) = \mathrm{rank}_{R/\mathfrak{p}}(M) \cdot \mathrm{rank}_{R/\mathfrak{q}}(N) \cdot \chi(R/\mathfrak{p}, R/\mathfrak{q})$
2. $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$, so that $\chi(M, N) = \mathrm{length}_R(M \otimes_R N) > 0$.

The first follows from the facts that (a) χ is additive on short exact sequences in each argument and (b) the Serre Vanishing Theorem: $\chi(M', N') = 0$ when $\mathrm{length}(M' \otimes_R N') < \infty$ and $\dim(M') + \dim(N') < \dim(R)$.

For the second, since M and N are MCM, we have $\mathrm{pd}_R(M) = \dim(R) - \dim(R/\mathfrak{p})$ and $\mathrm{pd}_R(N) = \dim(R) - \dim(R/\mathfrak{q})$, and thus by assumption $\mathrm{pd}_R(M) + \mathrm{pd}_R(N) = \dim(R)$. So, letting $F_\bullet$ and $G_\bullet$ be the minimal free resolutions of M and N, we have that $T_\bullet := F_\bullet \otimes_R G_\bullet$ lies in homological degrees 0 to $\dim(R)$, and it has finite length homology. The result follows from the claim:

> If R is CM and $0 \to T_{\dim(R)} \to \cdots \to T_1 \to T_0 \to 0$ is a finite free complex with finite length homology, then $H_i(T_\bullet) = 0$ for all $i > 0$.

This claim is easily proven using local cohomology, in a way similar to the proof of Proposition 5.4 below; see Remark 5.5. □

5. (Improved) New Intersection Theorem

Theorem 5.1 (NIT: New Intersection Theorem)**.** *If* $F_\bullet = (0 \to F_s \to \cdots \to F_0 \to 0)$ *is a finite free complex of* R*-modules with finite length homology and* $H_0(F_\bullet) \neq 0$ *then* $s \geq \dim(R)$*.*

This was proven in full generality by Paul Roberts. The mixed characteristic case involves an extremely complicated argument. (Greg Piepmeyer and I gave a "new" proof that is simpler.)

The following is equivalent to the Direct Summand Conjecture:

Theorem 5.2 (INIT: Improved New Intersection Theorem)**.** *Under the weaker assumptions that* $H_i(F_\bullet)$ *has finite length for all* $i \geq 1$ *and there is an element* $\alpha \in H_0(F_\bullet) \setminus \mathfrak{m}H_0(F_\bullet)$ *such that* $\mathfrak{m}^N\alpha = 0$ *for* $N \gg 0$*, we have* $s \geq \dim(R)$*.*

Clearly, $INIT \implies NIT$ (just take α to be any element of $H_0(F_\bullet) \setminus \mathfrak{m}H_0(F_\bullet)$). We also have

Proposition 5.3. *The New Intersection Theorem implies the Intersection Theorem.*

Proof. Assume $\text{length}(M \otimes_R N) < \infty$ and $\text{pd}_R(N) < \infty$. The goal is to show $\dim(M) \leq \text{pd}_R(N)$. First, observe that we may assume $M = R/I$ where $I = \text{ann}_R(M)$. Let $F_\bullet = (0 \to F_p \to \cdots \to F_0 \to 0)$ be the minimal free resolution of N over R and set $\overline{F}_\bullet = R/I \otimes_R F_\bullet$, a complex of free R/I-modules. Since $\text{supp}(R/I) \cap \text{supp}(N) = \{\mathfrak{m}\}$, the complex $\overline{F}_\bullet$ has finite length homology. Hence $p \geq \dim(R/I)$ by the NIT. □

Proposition 5.4. *Big MCM* $\Rightarrow$ *INIT*

Proof. We will cheat and only show Big MCM $\Rightarrow$ NIT carefully, and then just waive our hands a bit at INIT.

Suppose $F_\bullet$ has finite length homology, even in degree 0, and by way of contradiction, let us suppose $s < d = \dim(R)$. Let $\mathcal{C}^\bullet = (0 \to \mathcal{C}^0 \to \cdots \to \mathcal{C}^d \to 0)$ be the Cech complex on a sop $x_1, \ldots, x_d$ for R:

$$\mathcal{C}^\bullet = \left(0 \to R \to \bigoplus_i R[1/x_i] \to \bigoplus_{i<j} R[1/x_i, 1/x_j] \to \cdots \to R[1/x_1, \ldots, 1/x_d] \to 0\right).$$

Let B be a Big MCM R-module. We consider the totalization $T_\bullet$ of the bicomplex $F_\bullet \otimes_R B \otimes_R \mathcal{C}^\bullet$:

$$
\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & F_s \otimes B \otimes C^d & \longrightarrow & \cdots & \longrightarrow & F_0 \otimes B \otimes C^d & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & \vdots & & & & \vdots & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & F_s \otimes B \otimes C^1 & \longrightarrow & \cdots & \longrightarrow & F_0 \otimes B \otimes C^1 & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & F_s \otimes B \otimes C^0 & \longrightarrow & \cdots & \longrightarrow & F_0 \otimes B \otimes C^0 & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & 0 & &
\end{array}
$$

We proceed to calculate $H_0(T_\bullet)$ in two ways:

- The homology of the i-th column is $F_i \otimes_R H^*_{\mathfrak{m}}(B)$ and so, since B is MCM, each column only has homology in the top-most spot. Using that $s < d$, it follows that $H_0(T_\bullet) = 0$.
- Since $H_*(F_\bullet)$ has finite length, $F_\bullet[1/x]$ is exact for all $x \in \mathfrak{m}$. In particular, each row of this bicomplex, other than the bottom-most one, is exact. It follows that $T_\bullet \sim F_\bullet \otimes B \otimes C^0 = F_\bullet \otimes B$, and thus $H_0(T_\bullet) \cong H_0(F_\bullet) \otimes_R B$. This is non-zero since $\mathfrak{m}B \neq B$.

We have reached a contradiction, and this proves NIT.

For the INIT, the first calculation above remains valid: $H_0(T_\bullet) = 0$. With a bit more care, one can show that the assumptions regarding the element α are still enough to deduce that $H_0(T_\bullet) \neq 0$. $\square$

Remark 5.5. If we allow $s = d$ in this proof, the argument shows that $F. \otimes B$ only has homology in degree 0. In particular, suppose R is CM, so that we can take $B = R$. Given a finite free complex of the form $F_\bullet = (0 \to F_{\dim R} \to \cdots F_1 \to F_0 \to 0)$ that has finite length homology, we have $H_i(F_\bullet) = 0$ for $i > 0$.